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Pseudo-Triangle Visibility Graph: Characterizing and Reconstruction

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Pseudo-Triangle Visibility Graph

Characterizing and Reconstruction

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Abstract The visibility graph of a simple polygon represents visibility relations between its vertices. Knowing the correct order of the vertices around the boundary of a polygon and its visibility graph, it is an open problem to locate the vertices in a plane in such a way that it will be consistent with this visibility graph. This problem has been solved for special cases when we know that the target polygon is a *tower* or *spiral* one. In this paper, we propose a linear time reconstruction algorithm for a visibility graph which is assumed to belong to a simple polygon with at most three concave chains on its boundary, called a *pseuodo-triangle*. Moreover, we introduce a set of necessary and sufficient properties characterizing visibility graphs of pseudo-triangles and propose algorithms for checking these properties.

Keywords Computational geometry · Visibility graph · Characterizing visibility graph · Polygon reconstruction · Pseudo-triangle

1 Introduction

The *visibility graph* of a simple polygon $\mathcal{P}$ is a graph $\mathcal{G}(V, E)$ where V is the vertices of $\mathcal{P}$ and an edge (u, v) exists in E if and only if the line segment uv lies completely inside $\mathcal{P}$. In the visibility graph reconstruction problem, the goal is to build a polygon whose visibility graph is isomorphic to a given graph. *Everett* showed that this problem is in PSPACE[6], and this is the only result known for general polygons. This problem has been solved only for special cases of *spiral* and *tower* polygons when other than the visibility graph, the correct order of the vertices on the boundary of the polygon is known as well. These results are obtained by *Everett* and *Corneil* [7] for spiral polygons and by *Colley et al.* [3] for tower polygons. In spiral polygons there is at most one concave chain (Fig. 1a)

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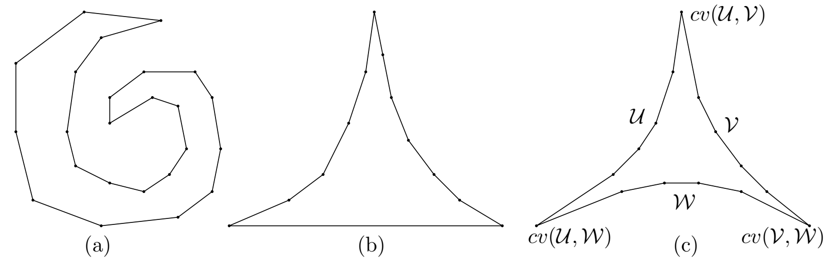


Fig. 1: (a) A Spiral polygon, (b) a tower polygon, and (c) a pseudo-triangle.

and the boundary of a tower polygon is composed of two concave chains and a single edge (Fig. 1b).

Although there is a bit progress on this type of reconstruction problem, there has been plenty of studies on characterizing visibility graphs [5,8,7,4,1,3]. In 1988, *Ghosh* introduced three necessary conditions for visibility graphs and conjectured their sufficiency[8]. In 1990, *Everett* proposed a counter-example graph rejecting *Ghosh*'s conjecture[6]. She also refined the third *Ghosh*'s necessary condition to a new stronger condition[9]. In 1992, *Abello et al.* built a graph satisfying *Ghosh*'s conditions and the stronger version of the third condition which was not the visibility graph of any simple polygon rejecting the sufficiency of these conditions[2]. In 1997, *Ghosh* added his fourth necessary condition and conjectured that this condition along with his first two conditions and the stronger version of the third condition are sufficient for a graph to be a visibility graph. Finally, in 2005 *Streinu* proposed a counter example for this conjecture[10].

In this paper, we solve the reconstruction problem for pseudo-triangles. A *pseudo-triangle* is a simple polygon consisting of three concave side-chains each pair shares one convex vertex (called a corner). Let $\mathcal{P}$ be a pseudo-triangle formed by the concave chains $\mathcal{U} = [cv(\mathcal{V},\mathcal{U}), \ldots, cv(\mathcal{U},\mathcal{W})]$, $\mathcal{V} = [cv(\mathcal{U},\mathcal{V}), \ldots, cv(\mathcal{V},\mathcal{W})]$, and $\mathcal{W} = [cv(\mathcal{U},\mathcal{W}), \ldots, cv(\mathcal{W},\mathcal{V})]$ where $cv(\mathcal{V},\mathcal{U}) = cv(\mathcal{U},\mathcal{V})$, $cv(\mathcal{U},\mathcal{W}) = cv(\mathcal{W},\mathcal{U})$, and $cv(\mathcal{W},\mathcal{V}) = cv(\mathcal{V},\mathcal{W})$ are the corners (Fig. 1c). We use $cv(\cdot,\cdot)$ as an abbreviation for the common vertex of the two side-chains.

Let $\mathcal{H} =< cv(\mathcal{V},\mathcal{U}), \ldots, cv(\mathcal{U},\mathcal{W}), \ldots, cv(\mathcal{W},\mathcal{V}), \ldots, cv(\mathcal{V},\mathcal{U}) >$ be the Hamiltonian cycle of the visibility graph of $\mathcal{P}$ which indicates the order of vertices on the boundary of $\mathcal{P}$. Knowing $\mathcal{H}$ for a visibility graph $\mathcal{G}(V,E)$, we introduce a set of necessary properties on $\mathcal{H}$ and $\mathcal{G}$ when this pair belongs to a pseudo-triangle and prove that these properties are sufficient as well.

Having these properties, we propose a linear-time algorithm for reconstructing a pseudo-triangle $\mathcal{P} =< cv(\mathcal{V},\mathcal{U}), \ldots, cv(\mathcal{U},\mathcal{W}), \ldots, cv(\mathcal{W},\mathcal{V}), \ldots, cv(\mathcal{V},\mathcal{U}) >$ with $\mathcal{G}(V,E)$ as its visibility graph. Moreover, we propose algorithms for verifying the properties on a given pair of $\mathcal{H}$ and $\mathcal{G}$. These characterizing algorithms run in linear time in terms of the size of $\mathcal{G}$. Therefore, in this paper we solve the characterizing and reconstructing problem for another class of polygons called pseudo-triangles.

Whileas a tower polygon is a special case of a pseudo-triangle, we use *Colley*'s algorithm as a sub-routine in our algorithm to build the initial part of the polygon.

Our motivation in solving this problem for pseudo-triangles is that every polygon can be partitioned into pseudo-triangles. To solve a general reconstruction problem, we can handle three steps:

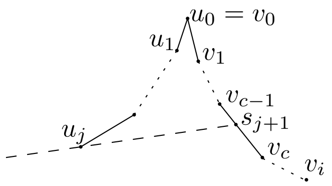


Fig. 2: Constructing a tower polygon.

- Recognize a pseudo-triangle decomposition for the target polygon from $\mathcal{G}(V, E)$ and $\mathcal{H}$.
- Reconstruct each pseudo-triangle separately.
- Attach the reconstructed pseudo-triangles supporting the pseudo-triangle decomposition and the visibility constraints.

In Section 2, we briefly describe *Colley*'s algorithm for reconstructing tower polygons which is used as a sub-routine in our algorithm. In Section 3, we describe the properties of the visibility graph of pseudo-triangles and in Section 4, we propose the reconstruction algorithm. Finally, we analyze the running time of algorithms required to check the properties and the reconstruction algorithm.

2 Reconstructing Tower Polygons

A *strong ordering* on a bipartite graph $\mathcal{G}(V, E)$ with partitions U and W is a pair of $<_U$ and $<_W$ orderings on respectively (*resp.*) U and W such that if $u <_U u'$, $w <_W w'$, and there are edges (u, w') and (u', w) in E, the edges (u, w) and (u', w') also exist in E.

The following theorem by *Colley et al.* [3] indicates the main property of the visibility graph of a tower polygon and guarantees the existence of a tower polygon consistent with such a visibility graph.

Theorem 1 *[3] Removing the edges of the reflex chains from the visibility graph of a tower gives an isolated vertex plus a connected bipartite graph for which the ordering of the vertices in the partitions provides a strong ordering. Conversely, any connected bipartite graph with strong ordering belongs to a tower polygon. Furthermore, such a tower can be constructed in linear time in terms of the number of vertices.*

The outline of *Colley*'s algorithm is as follows. As input, it takes the corner vertex $cv(\mathcal{U}, \mathcal{V}) = u_0 = v_0$ and a connected bipartite graph $\mathcal{G}(V, E)$ with vertices partitioned into two independent sets $\mathcal{U} = \{u_1, \ldots, u_m\}$ and $\mathcal{V} = \{v_1, \ldots, v_n\}$ having strong ordering. In the first step, the position of the corner $u_0 = v_0$ and the vertices u_1 and v_1 are determined as in Fig. 2. In a middle step, suppose that the positions of the vertices $cv(\mathcal{V}, \mathcal{U}), \ldots, u_{j-1}$ and $cv(\mathcal{U}, \mathcal{V}), \ldots, v_{k-1}$ are known and it has been determined the half-lines from u_{j-1} and v_{k-1} which contain u_j and v_k, respectively, where $(u_j, v_k) \in E$. To complete such a middle step, the position of the vertex u_j and the half-line from u_j which contains u_{j+1} (where u_{j+1} is visible from v_k) must be determined. For this purpose, u_j is located somewhere on its containing half-line horizontally below the vertex v_c which v_c has the minimum

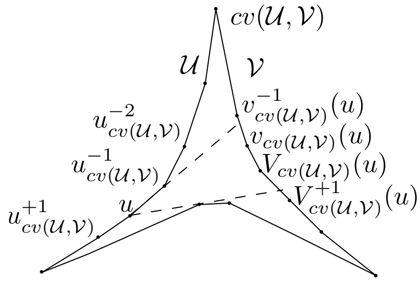


Fig. 3: Notations used for vertices.

index among vertices of $\mathcal{V}$ which are visible from u_{j+1}. Then, the containing half-line of u_{j+1} lies on the supporting line of u_j and s_{j+1} downward from u_j. Here, s_{j+1} is a point on $v_{c-1}v_c$ with an ϵ distance below s_j, when s_j lies on $v_{c-1}v_c$. If s_j does not lie on $v_{c-1}v_c$, then s_{j+1} is a point on $v_{c-1}v_c$ with an ϵ distance below v_{c-1}. According to this construction, s_j is the intersection of $\mathcal{V}$ and the supporting line of u_j and u_{j-1}. Similarly, r_i is the intersection of $\mathcal{U}$ and the supporting line of v_j and v_{j-1} (Fig. 2).

3 Properties of Pseudo-Triangle Visibility Graphs

In this section, we describe a set of properties that a pair of $\mathcal{H}$ and $\mathcal{G}$ must have to be the Hamiltonian cycle and visibility graph of a pseudo-triangle.

Any sub-sequence $< v_i, \ldots, v_j >$ on the Hamiltonian cycle is called a chain and is denoted by $[v_i, \ldots, v_j]$. A vertex v_a on a chain $[v_i, \ldots, v_j]$ is a *blocking vertex* for the invisible pair (v_i, v_j) if there is no visible pair of vertices v_l on $[v_i, \ldots, v_{a-1}]$ and v_k on $[v_{a+1}, \ldots, v_j]$. *Ghosh* showed that for every invisible pair of vertices (u, v) in a visibility graph, there is at least one blocking vertex in $[u, \ldots, v]$ or $[v, \ldots, u]$. Furthermore, every vertex on the shortest Euclidean path between u and v is a blocking vertex for this pair [8]. Note that In a pseudo-triangle the shortest Euclidean path between two invisible vertices turns in only one direction (i.e. clockwise or counterclockwise).

Let $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ be the side-chains of a pseudo-triangle. The order of vertices in these chains is defined with respect to one of their common vertices. Precisely, for a vertex $u \in \mathcal{U}$, $u^{-k}_{cv(\mathcal{U},\mathcal{V})}$ is a vertex in the subchain $[cv(\mathcal{U},\mathcal{V}), \ldots, u]$ where the length of the subchain $[u^{-k}_{cv(\mathcal{U},\mathcal{V})}, \ldots, u]$ is k. Similarly, $u^{+l}_{cv(\mathcal{U},\mathcal{V})}$ is a vertex in $[u, \ldots, cv(\mathcal{U},\mathcal{W})]$ where the length of $[u, \ldots, u^{+l}_{cv(\mathcal{U},\mathcal{V})}]$ is l. In addition, we use $u_{cv(\mathcal{U},\mathcal{V})}(x)$ and $U_{cv(\mathcal{U},\mathcal{V})}(x)$ to respectively denote the closest and the farthest vertices of $\mathcal{U}$ to $cv(\mathcal{U},\mathcal{V})$ that are visible to x where x lies on a side-chain other than $\mathcal{U}$. If $u' = u_{cv(\mathcal{U},\mathcal{V})}(x)$, then $u^{+i}_{cv(\mathcal{U},\mathcal{V})}(x)$ and $u^{-i}_{cv(\mathcal{U},\mathcal{V})}(x)$ notations are used for respectively $u'^{+i}_{cv(\mathcal{U},\mathcal{V})}$ and $u'^{-i}_{cv(\mathcal{U},\mathcal{V})}$. Similarly, if $u' = U_{cv(\mathcal{U},\mathcal{V})}(x)$, then $U^{+i}_{cv(\mathcal{U},\mathcal{V})}(x)$ and $U^{-i}_{cv(\mathcal{U},\mathcal{V})}(x)$ notations are used for respectively $u'^{+i}_{cv(\mathcal{U},\mathcal{V})}$ and $u'^{-i}_{cv(\mathcal{U},\mathcal{V})}$. Fig. 3 depicts these notations.

Lemma 1 *It is always possible to identify at least two corners of a pseudo-triangle $\mathcal{P}$ from its corresponding Hamiltonian cycle and visibility graph.*

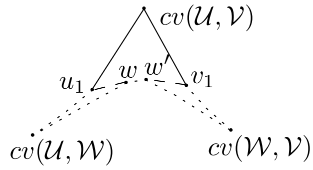


Fig. 4: A pseudo-triangle with $cv(\mathcal{U},\mathcal{W})$ and $cv(\mathcal{W},\mathcal{V})$ as its detectable corners.

Proof Since a corner is a convex vertex, it cannot be a blocking vertex for its neighbors. Therefore, in the Hamiltonian cycle of a pseudo-triangle, there are at most three vertices whose adjacent vertices are visible pairs. By traversing the Hamiltonian cycle, these visible pairs and so the corresponding corners can be identified.

Suppose that this method does not identify all three corners. Without loss of generality(*w.l.o.g.*), assume that $cv(\mathcal{U},\mathcal{V})$ is an unidentified corner. This means that $u_1 = cv(\mathcal{V},\mathcal{U})^{+1}_{cv(\mathcal{U},\mathcal{V})}$ and $v_1 = cv(\mathcal{U},\mathcal{V})^{+1}_{cv(\mathcal{U},\mathcal{V})}$ do not see each other and there must be a blocking vertex for this invisible pair. Due to their concavity, this blocking vertex cannot belong to the side-chains $\mathcal{U}$ and $\mathcal{V}$. Consider the Shortest Euclidean path between u_1 and v_1 (Fig. 4). It is clear that this path is a subchain of $\mathcal{W}$, saying $[w,\ldots,w']$ where $w' = w^{+l}_{cv(\mathcal{U},\mathcal{W})}$ and the edges (u_1,w) and (w',v_1) belong to the visibility graph. The polygon formed by $< u_1,\ldots,cv(\mathcal{U},\mathcal{W}),\ldots,w >$ is a tower polygon with base (u_1,w) and corner $cv(\mathcal{U},\mathcal{W})$. The corner of this tower is the isolated vertex obtained by removing the edges of its Hamiltonian cycle from its visibility graph. Therefore, the corner $cv(\mathcal{U},\mathcal{W})$ is detectable. The same argument holds for the tower polygon formed by $< w',\ldots,cv(\mathcal{W},\mathcal{V}),\ldots,v_1 >$ from which the corner $cv(\mathcal{W},\mathcal{V})$ can be identified. This means that if $cv(\mathcal{U},\mathcal{V})$ cannot be identified from the visibility graph, the other two corners will be detectable. □

Consider a pseudo-triangle $\mathcal{P}$ with side-chains $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$, and $\mathcal{G}$ and $\mathcal{H}$ as its visibility graph and Hamiltonian cycle, respectively. Assume that the method described in Lemma 1, identifies only two corners of $\mathcal{P}$. *W.l.o.g.*, assume that $cv(\mathcal{U},\mathcal{V})$ is the unidentified vertex. This means that there is a subchain on $\mathcal{W}$ which blocks the visibility of u_1 and v_1. Then, there is no visibility edge between a vertex from $\mathcal{U}$ and a vertex of $\mathcal{V}$. By removing the edges of the Hamiltonian cycle from the visibility graph, two isolated vertices $cv(\mathcal{U},\mathcal{W})$ and $cv(\mathcal{W},\mathcal{V})$ and a connected bipartite graph is obtained. By adding one of the adjacent edges (e) of these isolated vertices to this bipartite graph, we will have a single isolated vertex and a graph with strong ordering which according to Theorem 1 corresponds to a tower polygon with base e and $\mathcal{G}$ and $\mathcal{H}$ as its visibility graph and Hamiltonian cycle, respectively.

Therefore, we have the following property about the pair of $\mathcal{H}$ and $\mathcal{G}$ of a pseudo-triangle.

Property 1 *If $\mathcal{H}$ and $\mathcal{G}$ are respectively the Hamiltonian cycle and visibility graph of a pseudo-triangle $\mathcal{P}$, at least two corners of $\mathcal{P}$ can be identified. Furthermore, if only two corners are detectable, the given $\mathcal{H}$ and $\mathcal{G}$ belong to a pseudo-triangle if and only if there is a tower polygon with $\mathcal{H}$ and $\mathcal{G}$ as its Hamiltonian cycle and visiblity graph, respectively.*

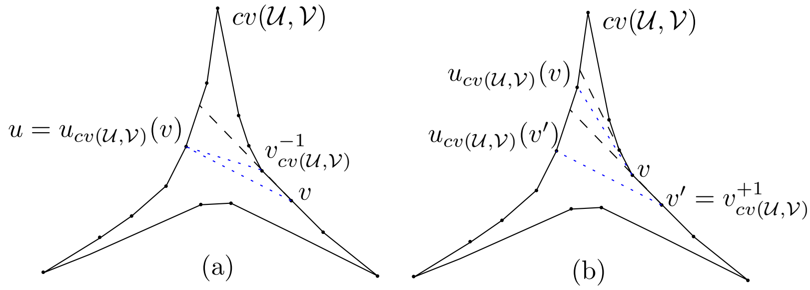


Fig. 5: (a) Property 3: $v^{-1}_{cv(\mathcal{U},\mathcal{V})}$ and $u_{cv(\mathcal{U},\mathcal{V})}(v)$ see each other, (b) Corollary 1: $u_{cv(\mathcal{U},\mathcal{V})}(v')$ cannot be closer to $cv(\mathcal{U},\mathcal{V})$ than $u_{cv(\mathcal{U},\mathcal{V})}(v)$.

So, in the remainder of this section we assume that the method described in the proof of Lemma 1 identifies all three corners.

An *interval* of a side-chain with endpoints p and q is the set of points on this side-chain connecting p to q. Note that it is not necessary for an endpoint of an interval to be a vertex of the side-chain.

Property 2 *Every non-corner vertex of a side-chain sees a single nonempty interval from each of the other side-chains.*

Proof The inner angle of such a vertex is more than π and its inner visibility region cannot be bounded by a single concave chain. Therefore, it will see some parts from any of the other side-chains. The continuity of this visible parts on each side-chain is proved by contradiction. Assume that a vertex $u \in \mathcal{U}$ sees two disjoint intervals $[v_i, \ldots, v_j]$ and $[v_k, \ldots, v_l]$ from $\mathcal{V}$ meaning that the interval $(v_j, \ldots, v_k)$ is not visible from u. Consider an invisible point v' in $(v_j, \ldots, v_k)$. There must be a blocking vertex for the invisible pair (u, v'). This blocking vertex must lie on the third side-chain which will also blocks either the visibility of u and v_j or u and v_k. □

Property 3 *(Fig. 5(a)) For any pair of side-chains $\mathcal{U}$ and $\mathcal{V}$ and a pair of vertices (u, v) where $u \in \mathcal{U}$, $v \in \mathcal{V}$, $v \neq cv(\mathcal{U},\mathcal{V})$, and $u = u_{cv(\mathcal{U},\mathcal{V})}(v)$, then $(v^{-1}_{cv(\mathcal{U},\mathcal{V})}, u) \in E$.*

Proof Consider the subpolygon $< u, u^{-1}_{cv(\mathcal{U},\mathcal{V})}, \ldots, cv(\mathcal{U},\mathcal{V}), \ldots, v^{-1}_{cv(\mathcal{U},\mathcal{V})}, v >$. If we triangulate this polygon, there is no internal diagonal connected to v which means that$< u, v, v^{-1}_{cv(\mathcal{U},\mathcal{V})} >$ must be a triangle in any triangulation. Therefore, the edge $(u, v^{-1}_{cv(\mathcal{U},\mathcal{V})})$ is a diagonal and this edge must exist in the visiblity graph. □

Corollary 1 *(Fig. 5(b)) For any pair of side-chains $\mathcal{U}$ and $\mathcal{V}$ and a vertex $v \in \mathcal{V}$, if $u_{cv(\mathcal{U},\mathcal{V})}(v) = u_j$ and $u_{cv(\mathcal{U},\mathcal{V})}(v^{+1}_{cv(\mathcal{U},\mathcal{V})}) = u_k$, then u_k is not closer than u_j to $cv(\mathcal{U},\mathcal{V})$.*

Corollary 2 *For any pair of side-chains $\mathcal{U}$ and $\mathcal{V}$ and a vertex $v \in \mathcal{V}$, if v does not see any vertex from $\mathcal{U}$, then $v^{+1}_{cv(\mathcal{U},\mathcal{V})}$ does not see any vertex of $\mathcal{U}$ as well.*

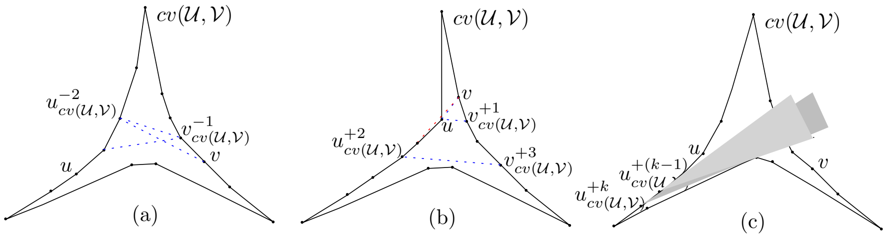


Fig. 6: (a) Property 4: $u^{-2}_{cv(\mathcal{U},\mathcal{V})}$ and $v^{-1}_{cv(\mathcal{U},\mathcal{V})}$ must see each other, (b) Property 5: u and $v^{+1}_{cv(\mathcal{U},\mathcal{V})}$ see each other, (c) Corollary 3: Visible vertices of $\mathcal{V}$ from $u^{+k}_{cv(\mathcal{U},\mathcal{V})}$ are also visible to $u^{+(k-1)}_{cv(\mathcal{U},\mathcal{V})}$.

Property 4 *(Fig. 6(a)) For any pair of side-chains $\mathcal{U}$ and $\mathcal{V}$ and a pair of vertices (u, v) where $u \in \mathcal{U}$ and $v \in \mathcal{V}$ and $k, l > 0$, if both $(u^{-k}_{cv(\mathcal{U},\mathcal{V})}, v)$ and $(u, v^{-l}_{cv(\mathcal{U},\mathcal{V})})$ exist in E, then $(u^{-k}_{cv(\mathcal{U},\mathcal{V})}, v^{-l}_{cv(\mathcal{U},\mathcal{V})})$ also exists in E.*

Proof If $u^{-k}_{cv(\mathcal{U},\mathcal{V})}$ and $v^{-l}_{cv(\mathcal{U},\mathcal{V})}$ do not see each other, then the blocking vertex cannot belong to the third side-chain. If this blocking vertex lies on $\mathcal{V}$, it will also block the visibility of $u^{-k}_{cv(\mathcal{U},\mathcal{V})}$ and v. Similarly, it cannot belong to $\mathcal{U}$. $\square$

Property 5 *(Fig. 6(b)) For any pair of side-chains $\mathcal{U}$ and $\mathcal{V}$ and a pair of vertices (u, v) where $u \in \mathcal{U}$ and $v \in \mathcal{V}$, if both $(u^{+k}_{cv(\mathcal{U},\mathcal{V})}, v^{+l}_{cv(\mathcal{U},\mathcal{V})})$ and (u, v) exist in E where $l, k > 0$, then at least one of the edges $(u^{+1}_{cv(\mathcal{U},\mathcal{V})}, v)$ and $(u, v^{+1}_{cv(\mathcal{U},\mathcal{V})})$ exists in E.*

Proof If follows from triangulating the subpolygon formed by boundary vertices $< u^{+k}_{cv(\mathcal{U},\mathcal{V})}, \ldots, u, v, \ldots, v^{+l}_{cv(\mathcal{U},\mathcal{V})} >$. $\square$

Corollary 3 *(Fig. 6(c)) For any pair of side-chains $\mathcal{U}$ and $\mathcal{V}$ and a pair of vertices $u \in \mathcal{U}$ and $v \in \mathcal{V}$, where $(u, v) \in E$ and none of the edges $(u^{+1}_{cv(\mathcal{U},\mathcal{V})}, v)$ and $(u, v^{+1}_{cv(\mathcal{U},\mathcal{V})})$ exists in E, all visible vertices of $\mathcal{V}$ from $u^{+k}_{cv(\mathcal{U},\mathcal{V})}$ are also visible from $u^{+(k-1)}_{cv(\mathcal{U},\mathcal{V})}$ (for any $k > 0$). This implies that $V_{cv(\mathcal{U},\mathcal{V})}(u^{+k}_{cv(\mathcal{U},\mathcal{V})})$ must lie above v.*

Proof Any visible vertex v' must belong to $[cv(\mathcal{U},\mathcal{V}), \ldots, v^{-1}_{cv(\mathcal{U},\mathcal{V})}]$. Otherwise, according to Property 5 either $(u^{+1}_{cv(\mathcal{U},\mathcal{V})}, v)$ or $(u, v^{+1}_{cv(\mathcal{U},\mathcal{V})})$ must exist. According to Corollary 1, $v_{cv(\mathcal{U},\mathcal{V})}(u)$ is closer to $cv(\mathcal{U},\mathcal{V})$ than $v_{cv(\mathcal{U},\mathcal{V})}(u^{+k}_{cv(\mathcal{U},\mathcal{V})})$, and because of the continuity of the chain visible from u (Property 2) , v' will be visible from u. This implies that v' is visible from all vertices of the chain $[u, \ldots, u^{+k}_{cv(\mathcal{U},\mathcal{V})}]$. $\square$

It is clear that every diagonal edge (u, v) in the visibility graph of a pseudo-triangle specifies a tower formed by the bounary vertices $< u, \ldots, cv(\mathcal{U},\mathcal{V}), \ldots, v >$. The vertices of this tower satisfy the strong ordering defined earlier. This strong ordering can be derived from properties 2, 4, and 5. Therefore, we do not specify this as a new property.

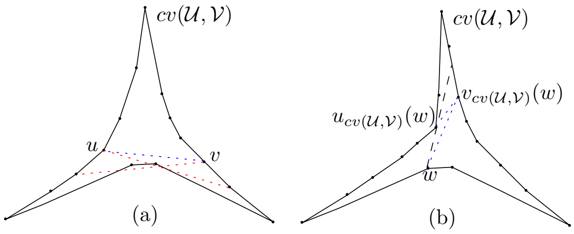


Fig. 7: (a) Property 6: u and v must see common vertices on mW, (b) Property 7: $u_{cv(\mathcal{U},\mathcal{V})}(w)$ and $v_{cv(\mathcal{U},\mathcal{V})}(w)$ must see each other.

Property 6 *(Fig. 7(a)) For any pair of side-chains $\mathcal{U}$ and $\mathcal{V}$ and a pair of vertices $u \in \mathcal{U}$ and $v \in \mathcal{V}$, where $(u, v) \in E$ and none of the edges $(u^{+1}_{cv(\mathcal{U},\mathcal{V})}, v)$ and $(u, v^{+1}_{cv(\mathcal{U},\mathcal{V})})$ exists in E, there is a nonempty subchain of the third side-chain $\mathcal{W}$ which is visible from both u and v.*

Proof Triangulating $\mathcal{P}$ using the edge (u, v), the adjacent triangle of this edge in the opposite side of $cv(\mathcal{U},\mathcal{V})$ must have its third vertex on $\mathcal{W}$. This is due to the invisibility of $(u^{+1}_{cv(\mathcal{U},\mathcal{V})}, v)$ and $(u, v^{+1}_{cv(\mathcal{U},\mathcal{V})})$ pairs. Therefore, this chain contains at least one vertex. From Property 2 we know that the visible part of $\mathcal{W}$ from any one of u and v vertices is continuous and their intersection will be continuous as well. □

Corollary 4 *For any side-chain $\mathcal{W}$, there exists at least one vertex $w \in \mathcal{W}$ that sees some vertices from both of the other side-chains. Furthermore, every vertex $w^{-k}_{cv(\mathcal{U},\mathcal{W})}$ where $k > 0$, sees at least one vertex from $\mathcal{U}$.*

Proof The first part follows from Property 6 and the second part follows from Property 3. □

Property 7 *(Fig. 7(b)) For any side-chain $\mathcal{W}$ and a vertex $w \in \mathcal{W}$ with distinct vertices $u = u_{cv(\mathcal{U},\mathcal{V})}(w)$ and $v = v_{cv(\mathcal{U},\mathcal{V})}(w)$, the vertices u and v are visible from each other.*

Proof Let $\mathcal{P}'$ be the subpolygon with $< cv(\mathcal{V},\mathcal{U}), \ldots, u, w, v, \ldots, cv(\mathcal{V},\mathcal{U}) >$ as its boundary vertices. The vertex w does not see any other vertex of $\mathcal{P}'$ which means that the diagonal uv must be used to triangulate $\mathcal{P}'$. This means that u and v must be visible from each other. □

Property 8 *(Fig. 8) For any pair of side-chains $\mathcal{U}$ and $\mathcal{V}$ and a pair of vertices $u \in \mathcal{U}$ and $v \in \mathcal{V}$, where $(u, v) \in E$ and none of the edges $(u^{+1}_{cv(\mathcal{U},\mathcal{V})}, v)$ and $(u, v^{+1}_{cv(\mathcal{U},\mathcal{V})})$ exists in E, let $[w, \ldots, w']$ be the maximum subchain of $\mathcal{W}$ visible to both u and v where $w' = w^{+l}_{cv(\mathcal{U},\mathcal{W})}$, $l \geq 0$. Then, w' is not closer to $cv(\mathcal{U},\mathcal{W})$ than $W_{cv(\mathcal{U},\mathcal{W})}(u^{+1}_{cv(\mathcal{U},\mathcal{V})})$.*

Proof According to Property 6, the subchain $[w, \ldots, w']$ is nonempty. For the sake of a contradiction, assume that $w'' = W_{cv(\mathcal{U},\mathcal{W})}(u^{+1}_{cv(\mathcal{U},\mathcal{V})})$ is farther from $cv(\mathcal{U},\mathcal{W})$

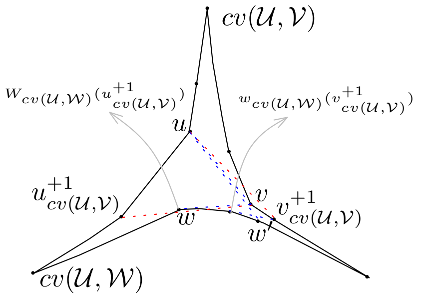


Fig. 8: Property 8: w' is not closer to $cv(\mathcal{U},\mathcal{W})$ than $W_{cv(\mathcal{U},\mathcal{W})}(u^{+1}_{cv(\mathcal{U},\mathcal{V})})$, Property 9: $w_{cv(\mathcal{U},\mathcal{W})}(v^{+1}_{cv(\mathcal{U},\mathcal{V})})$ is not closer to $cv(\mathcal{U},\mathcal{W})$ than $W_{cv(\mathcal{U},\mathcal{W})}(u^{+1}_{cv(\mathcal{U},\mathcal{V})})$.

than w'. Then, the edges $(w'', u^{+1}_{cv(\mathcal{U},\mathcal{V})})$ and (w', v) intersect within the pseudo-triangle. Let p be this intersection point. The subpolygon formed by the boundary vertices $< u, u^{+1}_{cv(\mathcal{U},\mathcal{V})}, p, v >$ is a convex polygon which completely lies inside the pseudo-triangle. So, the diagonal edge $(u^{+1}_{cv(\mathcal{U},\mathcal{V})}, v)$ must exist in E which is a contradiction. $\square$

Property 9 *(Fig. 8) For any pair of side-chains $\mathcal{U}$ and $\mathcal{V}$ and a pair of vertices $u \in \mathcal{U}$ and $v \in \mathcal{V}$, where $(u, v) \in E$ and none of the edges $(u^{+1}_{cv(\mathcal{U},\mathcal{V})}, v)$ and $(u, v^{+1}_{cv(\mathcal{U},\mathcal{V})})$ exists in E, $w_{cv(\mathcal{U},\mathcal{W})}(v^{+1}_{cv(\mathcal{U},\mathcal{V})})$ is not closer to $cv(\mathcal{U},\mathcal{W})$ than $W_{cv(\mathcal{U},\mathcal{W})}(u^{+1}_{cv(\mathcal{U},\mathcal{V})})$.*

Proof Let v' be $v^{+1}_{cv(\mathcal{U},\mathcal{V})}$ and u' be $u^{+1}_{cv(\mathcal{U},\mathcal{V})}$. For the sake of contradiction, assume that $w_{cv(\mathcal{U},\mathcal{W})}(v')$ is closer to $cv(\mathcal{U},\mathcal{W})$ than $W_{cv(\mathcal{U},\mathcal{W})}(u')$. Therefore, the edges $(v', w_{cv(\mathcal{U},\mathcal{W})}(v'))$ and $(u', W_{cv(\mathcal{U},\mathcal{W})}(u'))$ intersect within the pseudo-triangle. Let p be this intersection point. The subpolygon formed by the boundary vertices $< u, v, v', p, u' >$ is a convex polygon which completely lies inside the pseudo-triangle. So, all diagonal edges (u, v'), (u', v), and (u', v') must exist in E which is a contradiction. $\square$

Property 10 *(Fig. 9) For any side-chain $\mathcal{W}$, let u and v be respectively the closest vertices on $\mathcal{U}$ and $\mathcal{V}$ to $cv(\mathcal{U},\mathcal{V})$ which are visible from some vertex (not necessarily the same) of $\mathcal{W}$. Then, there exists a nonempty subchain $[w, \ldots, w']$ in $\mathcal{W}$, $w' = w^{+l}_{cv(\mathcal{U},\mathcal{W})}$ and $l \geq 0$, that either all vertices of this subchain are visible from both u and v, or, (w, w') is an edge of $\mathcal{W}$ and w sees v and w' sees u.*

Proof It is simple to show that $(u, v) \in E$. Assume that there is no vertex on $\mathcal{W}$ that sees both vertices u and v. Then, we first show that there is a pair of vertices w and $w' = w^{+l}_{cv(\mathcal{U},\mathcal{W})}$ where w sees v and w' sees u. Let w be $w_{cv(\mathcal{W},\mathcal{V})}(v)$ and w' be $w_{cv(\mathcal{U},\mathcal{W})}(u)$. Trivially, $w \neq w'$ and w is closer to $cv(\mathcal{U},\mathcal{W})$ than w' (otherwise, u and v will be visible to both w and w'). To complete the proof, it is enough to show that $w' = w^{+1}_{cv(\mathcal{U},\mathcal{W})}$. This is done by showing that any vertex w'' between w and w' on $\mathcal{W}$ must see at least one of the vertices u and v which contradicts the definition of w and w'.

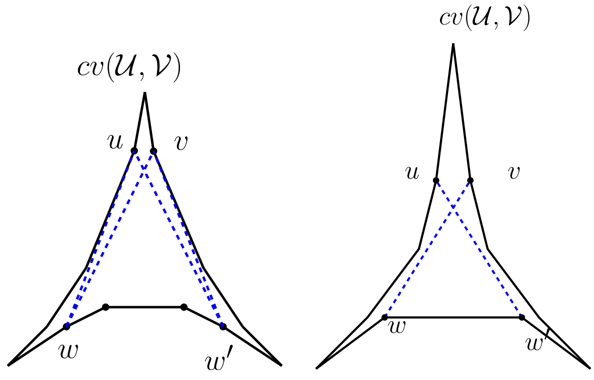


Fig. 9: Different cases of Property 10.

Assume that there is a vertex w'' between w and w' and it sees neither u nor v. In the tower polygon formed by boundary $< u, \ldots, cv(\mathcal{U},\mathcal{W}), \ldots, w'', w' >$, the blocking vertex for the invisible pair (w'', u) must lie on $\mathcal{U}$. Similarly, in the tower formed by boundary $< w, w'', \ldots, cv(\mathcal{W},\mathcal{V}), \ldots, v >$, the blocking vertex for the invisible pair (w'', v) must lie on $\mathcal{V}$. Therefore, at least one of the side-chains $\mathcal{U}$ and $\mathcal{V}$ must be convex which is a contradiction. So, w'' must see at least one of the vertices u and v. □

Corollary 5 *(Fig. 10) If w and w' satisfy the conditions of Property 10, then for $k > 0$:*

- $u_i = u_{cv(\mathcal{U},\mathcal{V})}(w'^{-k}_{cv(\mathcal{U},\mathcal{W})})$ *is not closer to* $cv(\mathcal{U},\mathcal{V})$ *than* $u_j = u_{cv(\mathcal{U},\mathcal{V})}(w'^{-(k-1)}_{cv(\mathcal{U},\mathcal{W})})$.
- *If there are vertices* $v_i = v_{cv(\mathcal{U},\mathcal{V})}(w^{-k}_{cv(\mathcal{U},\mathcal{W})})$ *and* $v_j = v_{cv(\mathcal{U},\mathcal{V})}(w^{-(k-1)}_{cv(\mathcal{U},\mathcal{W})})$*, then* v_i *is not closer to* $cv(\mathcal{U},\mathcal{V})$ *than* v_j.

These mean that as we move from w' to w the topmost visible vertices of $\mathcal{U}$ and $\mathcal{V}$ go down along these chains.

Proof For the sake of contradiction, assume that u_i is closer to $cv(\mathcal{U},\mathcal{V})$ than u_j. The diagonal edge $(w', u_{cv(\mathcal{U},\mathcal{V})}(w'))$ forms a tower polygon with boundary $< w', \ldots, cv(\mathcal{W},\mathcal{U}), \ldots, u_{cv(\mathcal{U},\mathcal{V})}(w') >$ which contains the vertices u_i and u_j, and satisfies strong ordering. Having the edges $(w'^{-k}_{cv(\mathcal{U},\mathcal{W})}, u_i)$ and $(w'^{-(k-1)}_{cv(\mathcal{U},\mathcal{W})}, u_j)$, the edge $(w'^{-(k-1)}_{cv(\mathcal{U},\mathcal{W})}, u_i)$ must also exist in E.

We prove the second part by contradiction. Let $w^{-l}_{cv(\mathcal{U},\mathcal{W})}$ be the closest vertex of $\mathcal{W}$ to $cv(\mathcal{U},\mathcal{W})$ which sees at least one vertex from $\mathcal{V}$ ($l \geq 0$). For $l \geq k > 0$, assume that v_i is closer to $cv(\mathcal{U},\mathcal{V})$ than v_j. Since $v_{cv(\mathcal{U},\mathcal{V})}(w)$ is not farther from $cv(\mathcal{U},\mathcal{V})$ than v_i, Property 4 implies that v_i sees w. According to Property 2, v_i is also visible from $w^{-(k-1)}_{cv(\mathcal{U},\mathcal{W})}$ which is a contradiction. □

Property 11 *Let $[w_i, \ldots, w_j]$ be the subchain of $\mathcal{W}$ satisfying Property 10 and for any vertex $w \in \mathcal{W}$, $u = u_{cv(\mathcal{U},\mathcal{V})}(w)$ and $v = v_{cv(\mathcal{U},\mathcal{V})}(w)$ are the closest vertices to $cv(\mathcal{U},\mathcal{V})$ which are visible to w. Then:*

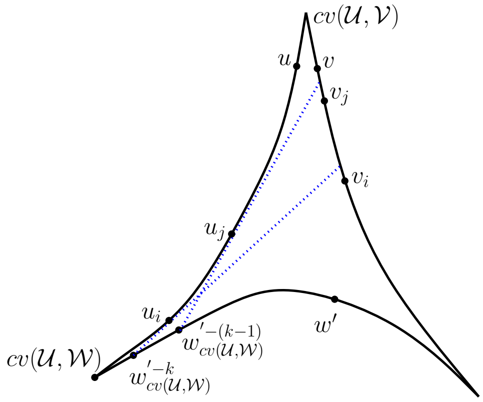


Fig. 10: Corollary 5: u_i (resp. v_i) is not closer to $cv(\mathcal{U},\mathcal{V})$ than u_j (resp. v_j).

- *If* $w \in [w_i, \ldots, w_j]$, *then at least one of the pairs* $(u^{+1}_{cv(\mathcal{U},\mathcal{V})}, v^{-1}_{cv(\mathcal{U},\mathcal{V})})$ *and* $(u^{-1}_{cv(\mathcal{U},\mathcal{V})}, v^{+1}_{cv(\mathcal{U},\mathcal{V})})$ *are invisible.*
- *If* $w \neq cv(\mathcal{U},\mathcal{W})$ *is closer to* $cv(\mathcal{U},\mathcal{W})$ *than* w_i, *then* $(u^{+1}_{cv(\mathcal{U},\mathcal{V})}, v^{-1}_{cv(\mathcal{U},\mathcal{V})})$ *is an invisible pair.*

Proof Consider the subpoygon $\mathcal{P}'$ with boundary $< w, v, \ldots, cv(\mathcal{U},\mathcal{V}), \ldots, u >$. The pairs $(w, v^{-1}_{cv(\mathcal{U},\mathcal{V})})$ and $(w, u^{-1}_{cv(\mathcal{U},\mathcal{V})})$ are invisible. These pairs share the same blocking vertex. If u is the blocking vertex, then $(u^{+1}_{cv(\mathcal{U},\mathcal{V})}, v^{-1}_{cv(\mathcal{U},\mathcal{V})})$ is an invisible pair, and if v is the blocking vertex, then the pair $(u^{-1}_{cv(\mathcal{U},\mathcal{V})}, v^{+1}_{cv(\mathcal{U},\mathcal{V})})$ is invisible.

Now, we show the correctness of the second part. It is clear that at least one of the vertices $u_{cv(\mathcal{U},\mathcal{V})}(w_i)$ and $v_{cv(\mathcal{U},\mathcal{V})}(w_i)$ is farther from $cv(\mathcal{U},\mathcal{V})$ than u and v. For the sake of contradiction, assume that $(u^{+1}_{cv(\mathcal{U},\mathcal{V})}, v^{-1}_{cv(\mathcal{U},\mathcal{V})})$ is a visible pair. Then, considering $\mathcal{P}' =< w, v, \ldots, cv(\mathcal{U},\mathcal{V}), \ldots, u >$, v must be the blocking vertex for the pairs $(w, v^{-1}_{cv(\mathcal{U},\mathcal{V})})$ and $(w, u^{-1}_{cv(\mathcal{U},\mathcal{V})})$. This vertex also blocks the pairs $(w^{+l}_{cv(\mathcal{U},\mathcal{W})}, u^{-i}_{cv(\mathcal{U},\mathcal{V})})$ and $(w^{+l}_{cv(\mathcal{U},\mathcal{W})}, v^{-j}_{cv(\mathcal{U},\mathcal{V})})$. But, for some $l > 0$ and i and $j \geq 0$, $w^{+l}_{cv(\mathcal{U},\mathcal{W})} = w'$, $u^{-i}_{cv(\mathcal{U},\mathcal{V})} = u_{cv(\mathcal{U},\mathcal{V})}(w')$, and $v^{-j}_{cv(\mathcal{U},\mathcal{V})} = v_{cv(\mathcal{U},\mathcal{V})}(w_i)$ which contradicts the definition of w_i. □

As it has been mentioned earlier, *Ghosh* introduced four necessary conditions for a visibility graph of a simple polygon. It is simple to show that these conditions are derived from the properties described in this section.

4 Pseudo-Triangle Reconstruction

In this section, $\mathcal{G}(V, E)$ denotes the visibility graph of a pseudo-triangle $\mathcal{P}$ with $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ side-chains and the order of vertices on the boundary of $\mathcal{P}$ is specified by a Hamiltonian cycle $\mathcal{H} =< cv(\mathcal{V},\mathcal{U}), \ldots, cv(\mathcal{U},\mathcal{W}), \ldots, cv(\mathcal{W},\mathcal{V}), \ldots, cv(\mathcal{V},\mathcal{U}) >$ in $\mathcal{G}$. We assume that the inputs $\mathcal{G}$ and $\mathcal{H}$ satisfy the properties 1 to 11. We propose an algorithm for reconstructing a pseudo-triangle corresponding to the given pair of $\mathcal{G}$ and $\mathcal{H}$.

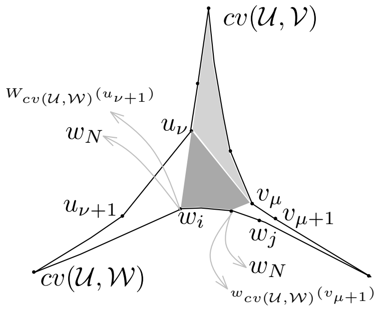


Fig. 11: The partitions of the initial polygon in reconstruction algorithm: the light-gray region is $\mathcal{X}$, the dark-gray is $\mathcal{Y}$ and the white parts are $\mathcal{Z}$ and $\mathcal{Z}'$.

In order to reconstruct the pseudo-triangle $\mathcal{P}$, we divide $\mathcal{P}$ into four subpolygons $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, and $\mathcal{Z}'$ as shown in Fig. 11 and reconstruct each one separately. For the sake of brevity, $u_i = cv(\mathcal{U},\mathcal{V})^{+i}_{cv(\mathcal{U},\mathcal{V})}$, $v_j = cv(\mathcal{U},\mathcal{V})^{+j}_{cv(\mathcal{U},\mathcal{V})}$, and $w_k = cv(\mathcal{U},\mathcal{W})^{+k}_{cv(\mathcal{U},\mathcal{W})}$ where $i,j,k \geq 0$. We assume that $\mathcal{U}$ and $\mathcal{V}$ have respectively $\alpha+1$ and $\delta+1$ vertices.

The subpolygon $\mathcal{X}$ is formed by subchains $[cv(\mathcal{V},\mathcal{U}),\dots,u_\nu]$ and $[cv(\mathcal{U},\mathcal{V}),\dots,v_\mu]$ and edge (u_ν,v_μ) where $V_{cv(\mathcal{U},\mathcal{V})}(u_\nu)=v_\mu$ and $U_{cv(\mathcal{U},\mathcal{V})}(v_\mu)=u_\nu$. The subpolygon $\mathcal{X}$ is a tower polygon with strong ordering in its visibility graph. Note that $u_{\nu+1}$ or $v_{\mu+1}$ exists only when the side-chain $\mathcal{W}$ has more than one edge, otherwise, two identified adjacent corners u_ν and v_μ compose the base of a tower polygon which can be constructed by *Colley*'s algorithm. So, we assume that $\mathcal{W}$ has more than one edge.

The subpolygon $\mathcal{Y}$ is identified as follows: Let $[w_i,\dots,w_j]$ be the maximum subchain of $\mathcal{W}$ visible from both u_ν and v_μ. According to Property 6, this chain is nonempty and continuous. Let $W_{cv(\mathcal{U},\mathcal{W})}(u_{\nu+1}) = w_k$ and $w_{cv(\mathcal{U},\mathcal{W})}(v_{\mu+1}) = w_l$. From Property 8, $k \leq j$ and $l \geq i$ and from Property 9, $k \leq l$. We define M and N as $\max(k,i)$ and $\min(l,j)$, respectively. It is clear that chain $[w_M,\dots,w_N]$ contains at least one vertex. Then, $\mathcal{Y}$ is defined to be the polygon with $< u_\nu, w_M, \dots, w_N, v_\mu >$ as its boundary.

The subpolygon $\mathcal{Z}$ is formed by subchains $[u_\nu,\dots,cv(\mathcal{U},\mathcal{W})]$ and $[cv(\mathcal{U},\mathcal{W}),\dots,w_M]$ and edge (u_ν,w_M). Similarly, subchains $[v_\mu,\dots,cv(\mathcal{V},\mathcal{W})]$ and $[cv(\mathcal{V},\mathcal{W}),\dots,w_N]$ and edge (v_μ,w_N) specify the subpolygon $\mathcal{Z}'$. It is clear that $\mathcal{P}$ is the union of $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, and $\mathcal{Z}'$.

Our reconstruction algorithm first builds $\mathcal{X}$ using *Colley*'s algorithm in such a way that vertices of $\mathcal{U}$ lie to the left of vertices of $\mathcal{V}$. According to this algorithm, positions of vertices $cv(\mathcal{V},\mathcal{U}),\dots,u_{\nu-1}$ and $cv(\mathcal{U},\mathcal{V}),\dots,v_{\mu-1}$ are obtained and directions of the edges $(u_{\nu-1},u_\nu)$ and $(v_{\mu-1},v_\mu)$ are determined. In order to specify the actual position of u_ν, choose a point on its half-line that is below the horizontal line passing through $v_{cv(\mathcal{U},\mathcal{V})}(u_\nu)$. The position of v_μ is determined similarly. Then, we extend this polygon to build $\mathcal{Y}$ (Section 4.1) and build and attach $\mathcal{Z}$ and $\mathcal{Z}'$ parts to this polygon (Section 4.2) to complete the construction procedure.

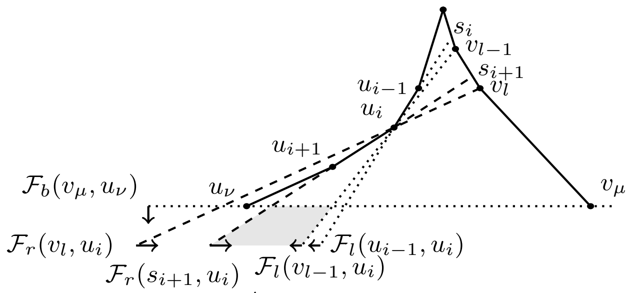


Fig. 12: $\mathscr{W}^j_{i,l}$ is the shaded region.

4.1 Reconstructing $\mathcal{Y}$

In this step, we build the subpolygon $\mathcal{Y} =< u_\nu, w_M, \ldots, w_N, v_\mu >$. We know the position of vertices u_ν and v_μ from the previous step, which are also on the boundary of $\mathcal{Y}$. To locate positions of other vertices, we show that there are nonempty regions in which these vertices can be placed.

For an arbitrary vertex w_j from $\mathcal{Y}$ which $u_{cv(\mathcal{U},\mathcal{V})}(w_j) = u_i$ and $v_{cv(\mathcal{U},\mathcal{V})}(w_j) = v_l$, we define a region $\mathscr{W}^j_{i,l}$ from which each point sees all vertices in the subchains $[u_i, \ldots, u_\nu]$ and $[v_l, \ldots, v_\mu]$. Therefore, w_j can be placed in $\mathscr{W}^j_{i,l}$ supporting the visibility constraints between w_i and vertices of $\mathcal{X}$. We use $\mathscr{W}^j$ instead of $\mathscr{W}^j_{i,l}$ whenever i and l indices are not important. The region $\mathscr{W}^j$ is determined as follows: While w_j sees u_ν and v_μ, the vertices u_i and v_l always exist and are well-defined. If u_i and v_l are identical, then $i = l = 0$ and the region $\mathscr{W}^j = \mathscr{W}^j_{0,0}$ is defined to be the part of the cone formed by the lines through $(cv(\mathcal{V},\mathcal{U}), u_1)$ and $(cv(\mathcal{V},\mathcal{U}), v_1)$ restricted to the underneath of the line through $u_\nu v_\mu$. Trivially, each point of $\mathscr{W}^j$ sees all vertices $u_\nu, \ldots, cv(\mathcal{V},\mathcal{U}) = cv(\mathcal{U},\mathcal{V}), \ldots, v_\mu$.

Let $\mathcal{F}_z(x, y)$ be the 'z' half-plane defined by the line through x and y where 'z' is 'b' (bottom), 'r' (right), or 'l' (left). If u_i and v_l are distinct vertices, according to Property 11, at least one of the pairs (u_{i+1}, v_{l-1}) and (u_{i-1}, v_{l+1}) do not see each other. The invisible pair is determined by applying Corollary 5 and Property 11. *W.l.o.g.*, assume that (u_{i+1}, v_{l-1}) is the invisible pair. Then, $\mathscr{W}^j_{i,l}$ is defined to be $\mathcal{F}_r(s_{i+1}, u_i) \bigcap \mathcal{F}_r(v_l, u_i) \bigcap \mathcal{F}_l(u_{i-1}, u_i) \bigcap \mathcal{F}_l(v_{l-1}, u_i) \bigcap \mathcal{F}_b(v_\mu, u_\nu)$ (Fig. 12). It is simple to see that all points in this region satisfy the visibility constraints of w_j with respect to the vertices $u_\nu, \ldots, cv(\mathcal{U},\mathcal{V}), \ldots, v_\mu$.

From concavity of $\mathcal{U}$ and $\mathcal{V}$, $\mathcal{F}_r(s_{i+1}, u_i) \bigcap \mathcal{F}_l(u_{i-1}, u_i)$ and $\mathcal{F}_r(v_l, u_i) \bigcap \mathcal{F}_l(v_{l-1}, u_i)$ are not empty. Therefore, $\mathscr{W}^j_{i,l}$ will be empty only when $\mathcal{F}_r(s_{i+1}, u_i) \bigcap \mathcal{F}_l(v_{l-1}, u_i)$ is empty or $\mathcal{F}_r(v_l, u_i) \bigcap \mathcal{F}_l(u_{i-1}, u_i)$ is empty. The first case is impossible, because otherwise, u_{i+1} must be visible from v_{l-1} which is in contradiction with invisibility assumption of (u_{i+1}, v_{l-1}). The second case is also impossible, because then, the pair u_i and v_l must be invisible. But, according to Property 7, u_i and v_l must be visible from each other.

Therefore, the region $\mathcal{F}_r(s_{i+1}, u_i) \bigcap \mathcal{F}_r(v_l, u_i) \bigcap \mathcal{F}_l(u_{i-1}, u_i) \bigcap \mathcal{F}_l(v_{l-1}, u_i)$ is nonempty and some part of this intersection lies in half-plane $\mathcal{F}_b(v_\mu, u_\nu)$.

According to the above discussion, $\mathscr{W}^j$ is defined by $\mathcal{F}_b(v_\mu, u_\nu)$ and two half-planes of $\{\mathcal{F}_r(s_{i+1}, u_i), \mathcal{F}_r(v_l, u_i), \mathcal{F}_l(u_{i-1}, u_i), \mathcal{F}_l(v_{l-1}, u_i)\}$. The apex of $\mathscr{W}^j$ is defined to be the intersection of the corresponding lines of these two half-planes.

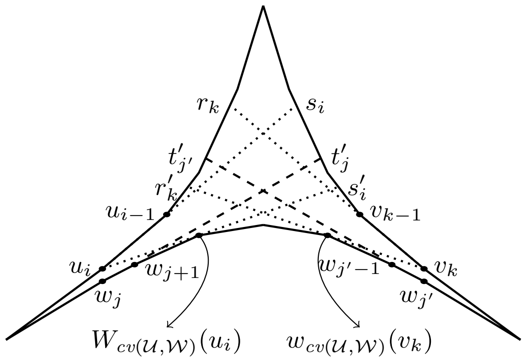


Fig. 13: Points $s_{(\cdot)}$, $s'_{(\cdot)}$, $r_{(\cdot)}$, $r'_{(\cdot)}$, and $t'_{(\cdot)}$.

Note that if the apex of $\mathscr{W}^j$ lies on $\mathcal{U}$, the apex of $\mathscr{W}^{j-1}$ will lie on $\mathcal{U}$ as well. Furthermore, $\mathscr{W}^{j-1}$ is either completely coinsiding $\mathscr{W}^j$ or is completely on its left. Similarly, if the apex of $\mathscr{W}^j$ lies on $\mathcal{V}$, then the apex of $\mathscr{W}^{j+1}$ lies on $\mathcal{V}$ as well, and $\mathscr{W}^{j+1}$ is either coinciding $\mathscr{W}^j$ or is completely on its right.

Then, we can place the vertices $w_M, \ldots, w_N$ of $\mathcal{Y}$ on an arbitrary concave chain inside $\mathcal{F}_b(v_\mu, u_\nu)$ in such a way that $w_j \in \mathscr{W}^j$. This placement satisfies the visibility constraints for $\mathcal{X}$ and $\mathcal{Y}$. However, to guarantee the reconstruction of $\mathcal{Z}$ and $\mathcal{Z}'$, we define some constraints on this concave chain which is described in the rest of this section.

Let s'_i $(i > \nu)$ be the intersection of $\mathcal{V}$ and the line through u_i and $W_{cv(\mathcal{U},\mathcal{W})}(u_i)$, r'_k $(k > \mu)$ be the intersection of $\mathcal{U}$ and the line through v_k and $w_{cv(\mathcal{U},\mathcal{W})}(v_k)$, t'_j $(j < M)$ be the intersection of $\mathcal{V}$ and the line through w_j and w_{j+1}, and t'_j $(j > N)$ be the intersection of $\mathcal{U}$ and the line through w_j and w_{j-1} (see Fig. 13).

Note that although we have not yet determined positions of vertices defining s'_i, r'_k, and t'_j, we can determine their containing edges from the visibility information as follows: for $i > \nu$, if u_i sees at least one vertex from $\mathcal{V}$, s_i lies on $(v^{-1}_{cv(\mathcal{U},\mathcal{V})}(u_i), v_{cv(\mathcal{U},\mathcal{V})}(u_i))$ and s'_i lies on $(V_{cv(\mathcal{U},\mathcal{V})}(u_i), V^{+1}_{cv(\mathcal{U},\mathcal{V})}(u_i))$. On the other hand, if u_i sees no vertex from $\mathcal{V}$, then for $k \geq i$, both s_k and s'_k lie on $(V^{-1}_{cv(\mathcal{U},\mathcal{V})}(u_j), V_{cv(\mathcal{U},\mathcal{V})}(u_j))$ where u_j has the highest index among the vertices of $\mathcal{U}$ that see at least one vertex from $\mathcal{V}$. However, this has an exception when $i = \nu+1$ and w_{M-1} is visible to both u_ν and v_μ, for which both s_k and s'_k for $k \geq i$ lie on $(v_\mu, v_{\mu+1})$. The same situation happens for r_l and r'_l when $l > \mu$.

The containing edge of t'_j for $j < M$ is determined as follows: If w_j sees at least one vertex from $\mathcal{V}$, then t'_j lies on $(V_{cv(\mathcal{U},\mathcal{V})}(w_j), V^{+1}_{cv(\mathcal{U},\mathcal{V})}(w_j))$, otherwise, it lies on the containing edge of s'_α (Note that according to our assumption at the beginning of Section 4, α and δ are respectively the greatest indices of vertices u_i and v_j on $\mathcal{U}$ and $\mathcal{V}$ side-chains.) Similarly for $j > N$, if w_j sees at least one vertex from $\mathcal{U}$, then t'_j lies on $(U_{cv(\mathcal{U},\mathcal{V})}(w_j), U^{+1}_{cv(\mathcal{U},\mathcal{V})}(w_j))$, and otherwise, it lies on the containing edge of r'_δ.

The containing edges of s'_α and r'_δ are respectively called *"the floating edge in* $\mathcal{V}$*"* and *"the floating edge in* $\mathcal{U}$*"*. We call these edges floating because we increase their length, and reposition their underneath vertices to enforce the concavity in building $\mathcal{Z}$ and $\mathcal{Z}'$.

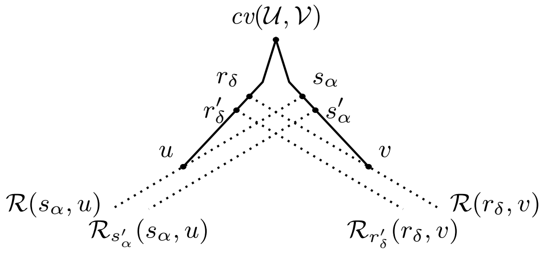


Fig. 14: The rays $\mathcal{R}(s_\alpha,u)$, $\mathcal{R}_{s'_\alpha}(s_\alpha,u)$, $\mathcal{R}(r_\delta,v)$, and $\mathcal{R}_{r'_\delta}(r_\delta,v)$.

We define the vertices w_{M^*} and w_{N^*} as follows: If $\mathcal{W}$ has two edges, then w_M and w_N are both equal to w_1 (the middle vertex of $\mathcal{W}$), and w_{M^*} and w_{N^*} are also defined to be w_1. When $\mathcal{W}$ has more than two edges, M^* is defined to be M when the apex of $\mathscr{W}^M$ does not lie on a vertex of $\mathcal{V}$ below its floating edge. Otherwise, M^* is defined to be j where j is the maximum index for which the apex of $\mathscr{W}^j$ lies above the floating edge of $\mathcal{V}$ (this edge may lie on $\mathcal{U}$). If the index of $u_{cv(\mathcal{U},\mathcal{V})}(w_{M^*})$ is greater than ν, the apex of $\mathscr{W}^{M^*}$ is temporarily assumed to be u_ν and $\mathscr{W}^{M^*}$ is defined to lie between $\mathcal{F}_r(s_{i+1},u_\nu)$ and $\mathcal{F}_l(u_{\nu-1},u_\nu)$. The index N^* is defined similarly. It is clear that at least $w_{M^*}=w_M$ or $w_{N^*}=w_N$.

We use $\mathcal{R}(x,y)$ to denote the ray from x towards y. In addition, $\mathcal{R}_a(x,y)$ denotes the ray from a and parallel to $\mathcal{R}(x,y)$ (Fig. 14).

Despite our definition of the regions $\mathscr{W}^i$ for all vertices $w_i \in \mathcal{W}$, we refine this definition for $\mathscr{W}^{N^*}$ (*resp.* $\mathscr{W}^{M^*}$) when $N^* \neq N$ (*resp.* $M^* \neq M$) or the floating edge of $\mathcal{V}$ (*resp.* $\mathcal{U}$) lies under the line through u_ν and d_μ. It is simple to see that at most one of the floating edges lies under $\mathcal{R}(u_\nu,d_\mu)$. Let v be a point on $\mathcal{R}_{v_\mu}(r_{\mu+1},v_\mu)$ when the floating edge of $\mathcal{V}$ lies under $\mathcal{R}(u_\nu,v_\mu)$, or be v_μ otherwise. Similarly, u is defined to be either u_ν or a point on $\mathcal{R}_{u_\nu}(s_{\nu+1},u_\nu)$. The regions $\mathscr{W}^{N^*}$ and $\mathscr{W}^{M^*}$ are restricted to lie under the line through u and v. Moreover, we know that at most one of the indices M^* and N^* is not equal to its corresponding index M or N. *W.l.o.g*, assume that $N^* \neq N$. Then, we additionally restrict the region $\mathscr{W}^{N^*}$ as follows (this restriction is not applied when we reconstruct $\mathcal{Z}$ or $\mathcal{Z}'$). Let p be a point inside the intersection of $\mathscr{W}^N$ and $\mathcal{F}_b(u,v)$ and with an arbitrary positive distance from $\mathcal{R}(u,v)$. We determine t'_{N^*} on its edge and with ϵl distance above the lower endpoint of this edge where $\epsilon > 0$ and l is the number of vertices in $\mathcal{V}$ and $\mathcal{W}$ whose $r'_{(\cdot)}$'s and $t'_{(\cdot)}$'s lie on this edge. The region $\mathscr{W}^{N^*}$ is restricted to lie under the line through t'_{N^*} and p (see Fig. 15).

Let s_α be a point on its edge and with ϵk distance below the upper endpoint of this edge where $\epsilon > 0$ and k is the number of vertices in $\mathcal{U}$ whose $s_{(\cdot)}$'s lie on this edge. Similarly, let s'_α be a point on its edge and with ϵm distance above the lower endpoint of this edge where $\epsilon > 0$ and m is the number of vertices in $\mathcal{U}$ and $\mathcal{W}$ whose $s'_{(\cdot)}$'s and $t'_{(\cdot)}$'s lie on this edge. The value of ϵ is small enough such that s_α lies above s'_α. The points r_δ and r'_δ are defined similarly.

Let $\mathcal{S}$ (*resp.* $\mathcal{T}$) be the strip defined by the supporting lines of $\mathcal{R}(s_\alpha,u)$ and $\mathcal{R}_{s'_\alpha}(s_\alpha,u)$ (*resp.* $\mathcal{R}(r_\delta,v)$ and $\mathcal{R}_{r'_\delta}(r_\delta,v)$).

Lemma 2 *It is always possible to enlarge the floating edges of $\mathcal{V}$ (resp. $\mathcal{U}$) such that the intersection of $\mathscr{W}^{M^*}$ (resp. $\mathscr{W}^{N^*}$) and $\mathcal{S}$ (resp. $\mathcal{T}$) is not empty.*

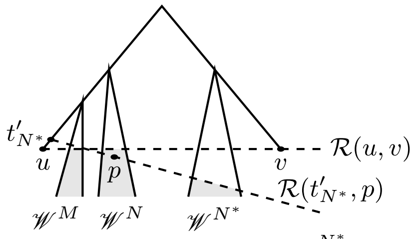


Fig. 15: Restricting $\mathscr{W}^{N^*}$.

Proof Assume that the intersection of $\mathscr{W}^{M^*}$ and $\mathcal{S}$ is empty. According to the definition of M^*, the apex of $\mathscr{W}^{M^*}$ either lies above the floating edge of $\mathcal{V}$ or lies on $\mathcal{U}$. This implies that enlarging the floating edge of $\mathcal{V}$ only affects the upper half-plane that defines $\mathscr{W}^{M^*}$. Then, we can enlarge the floating edge of $\mathcal{V}$ in such a way that the lower defining ray of $\mathcal{S}$ and the upper defining half-plane of $\mathscr{W}^{M^*}$ intersect inside $\mathscr{W}^{M^*}$ which means that the intersection of $\mathscr{W}^{M^*}$ and $\mathcal{S}$ is not empty. Moreover, when this intersection is not empty, this extension will just increase the intersection. To complete the proof, it is simple to see that extending the floating edge of $\mathcal{U}$ will again increase the intersection of $\mathcal{W}^{M^*}$ and $\mathcal{S}$.

The proof for w_{N^*} is analogously the same. □

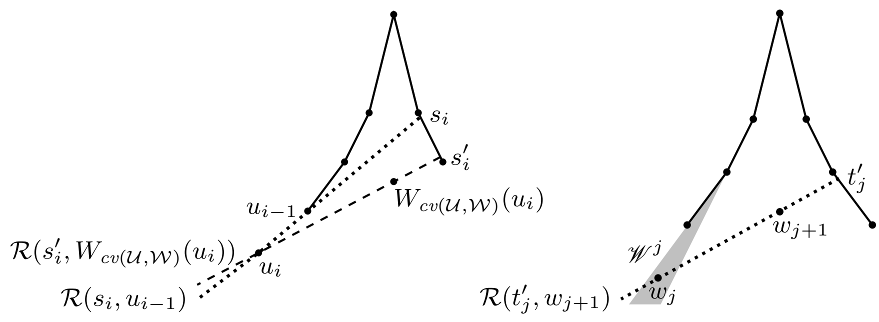
After locating the position of vertices in $\mathcal{X}$ (by possibly extending the floating edges), we place the vertices of $\mathcal{Y}$ as follows: If $N^* \neq N$, then we set p as w_N and place $w_{M^*} = w_M$ inside the intersection of $\mathscr{W}^M$ and $\mathcal{S}$ in such a way that both w_M and w_N be visible to u and v; neither w_M blocks the visibility of w_N, nor w_N blocks the visibility of w_M. When $M^* \neq M$, w_M and w_N are positioned analogously. Finally, if $M^* = M$ and $N^* = N$, we select a point from $\mathcal{S} \cap \mathscr{W}^M$ as w_M and a point from $\mathcal{T} \cap \mathscr{W}^N$ as w_N again in such a way that both see u and v. Then, we put the vertices $w_{M+1}, \ldots, w_{N-1}$ on a slightly concave chain from w_M to w_N in such a way that each w_j ($M \leq j \leq N$) lies inside $\mathscr{W}^j$ and sees u and v. It is simple to check that this setting is compatible with the visibility graph restricted to the vertices of $\mathcal{X}$ and $\mathcal{Y}$.

4.2 Reconstructing $\mathcal{Z}$ and $\mathcal{Z}'$

In this step, we place the vertices of $\mathcal{Z}$ and $\mathcal{Z}'$ to complete the reconstruction procedure. As said before, $\mathcal{Z}$ (resp. $\mathcal{Z}'$) is a part of the target pseudo-triangle with $< u_\nu, u_{\nu+1}, \ldots, cv(\mathcal{U}, \mathcal{W}), \ldots, w_M >$ (resp. $< v_\mu, v_{\mu+1}, \ldots, cv(\mathcal{V}, \mathcal{W}), \ldots, w_N >$) boundary vertices. Here, we only describe how to build $\mathcal{Z}$. The construction of $\mathcal{Z}'$ is the same.

Location of a vertex $u_i \in \mathcal{Z}$ is determined by the intersection point of the rays $\mathcal{R}(s_i, u_{i-1})$ and $\mathcal{R}(s'_i, W_{cv(\mathcal{U},\mathcal{W})}(u_i))$ and location of a vertex $w_h \in \mathcal{Z}$ is an arbitrary point on $\mathcal{R}(t'_h, w_{h+1})$ inside the region $\mathscr{W}^h$. Therefore, to construct $\mathcal{Z}$ we start from $u_{\nu+1}$ and w_{M-1}, and in each step we determine the position of one of the vertices and go forward to the next vertex. This is done by incrementally determining direction of the rays $\mathcal{R}(s_i, u_{i-1})$, $\mathcal{R}(s'_i, W_{cv(\mathcal{U},\mathcal{W})}(u_i))$, and $\mathcal{R}(t'_h, w_{h+1})$ as well as $\mathscr{W}^h$ regions.

Consider the edges of the pseudo-triangle on which the points s_i, s'_i, r_j, r'_j, and t'_l for $i > \nu$, $j > \mu$, and $l < M$ and $l > N$ lie. Keep an upper point and a



(a) (b)

Fig. 16: (a) Determining u_i, (b) Determining w_j.

lower point for each edge. Initialize the upper point with the upper endpoint of that edge or the latest located $s_{(\cdot)}$ or $r_{(\cdot)}$ on this edge. Initialize the lower point with the lower endpoint of the edge. Position of each $s_{(\cdot)}$, $r_{(\cdot)}$, $s'_{(\cdot)}$, $r'_{(\cdot)}$, and $t'_{(\cdot)}$ is determined whenever we need the rays passing through them. We place the points $s'_{(\cdot)}$, $r'_{(\cdot)}$, and $t'_{(\cdot)}$, with $\epsilon > 0$ distance above the current lower point of their edges and place the points $s_{(\cdot)}$ and $r_{(\cdot)}$, with $\epsilon > 0$ distance below the upper point of their edges. Whenever a new $s_{(\cdot)}$, $r_{(\cdot)}$, $s'_{(\cdot)}$, $r'_{(\cdot)}$, or $t'_{(\cdot)}$ point is located on an edge, the upper or lower point of that edge is updated properly.

More precisely, assume that we have already determined positions of vertices $u_\nu, u_{\nu+1}, \ldots, u_{i-1}$ ($i > \nu$) as well as the vertices $w_M, w_{M-1}, \ldots, w_{j+1}$ ($j < M$). To determine position of one of the vertices u_i and w_j we do as follows: Let w_k be $W_{cv(\mathcal{U},\mathcal{W})}(u_i)$. If $k < j$, then we have already located the position of w_k, and directions of the rays $\mathcal{R}(s_i, u_{i-1})$ and $\mathcal{R}(s'_i, W_{cv(\mathcal{U},\mathcal{W})}(u_i))$ are known. We will show in Lemma 3 that these rays intersect. So, u_i is located on the intersection point of these rays (Fig. 16a). Otherwise, we must first determine position of w_j which lies on $\mathcal{R}(t'_j, w_{j+1})$ and inside $\mathscr{W}^j$ (Fig. 16b). The position of w_{j+1} is already known and t'_{j+1} is determined according to the above paragraph. From these two points the direction of $\mathcal{R}(t'_j, w_{j+1})$ is obtained. The region $\mathscr{W}^j$ is determined as follows: Suppose that $u_{cv(\mathcal{U},\mathcal{V})}(w_j) = u_k$ and $v_{cv(\mathcal{U},\mathcal{V})}(w_j) = v_l$. We define $\mathscr{W}^j$ as in the previous section with the exception that it may be possible that only one of the vertices u_k and v_l exists. By Corollary 4, for $j < M$, u_j always exists. If w_j sees no vertex from $\mathcal{V}$, then it would see a part of the floating edge of $\mathcal{V}$. Hence, we consider the upper endpoint of this edge as v_{l-1}. From properties 7 and 11 we know that $\mathscr{W}^j$ is not empty and lies to the left of $\mathscr{W}^{j+1}$. Moreover, it will be shown in Lemma 3 that $\mathcal{R}(t'_j, w_{j+1})$ intersects $\mathcal{R}_{s'_\alpha}(s_\alpha, u)$. Since $\mathcal{R}_{s'_\alpha}(s_\alpha, u)$ passes through all $\mathscr{W}^{(\cdot)}$, $\mathcal{R}(t'_j, w_{j+1})$ passes through $\mathscr{W}^j$. Therefore, we can determine the position of w_j. According to the definition of $\mathcal{R}(s_i, u_{i-1})$ and $\mathcal{R}(s'_i, W_{cv(\mathcal{U},\mathcal{W})}(u_i))$ for u_i and $\mathcal{R}(t'_j, w_{j+1})$ and $\mathscr{W}^j$ for w_j, in both cases (locating u_i or w_j), visibility of the newly located vertex is exactly the same as its visibility in the visibility graph (restricted to the vertices of $\mathcal{X}$, $\mathcal{Y}$, and the constructed part of $\mathcal{Z}$).

Lemma 3 *The rays $\mathcal{R}(s_i, u_{i-1})$ and $\mathcal{R}(s'_i, W_{cv(\mathcal{U},\mathcal{W})}(u_i))$ for $i > \nu$ are convergent inside $\mathcal{S}$.*

Proof Remember that $\mathcal{S}$ is the strip defined by the supporting lines of $\mathcal{R}(s_\alpha, u)$ and $\mathcal{R}_{s'_\alpha}(s_\alpha, u)$. By Corollary 1 and Corollary 3, we know that s_i lies above the strip $\mathcal{S}$ and s'_i lies below this strip. Then, it is enough to show that for $i > \nu$, $\mathcal{R}(s'_i, W_{cv(\mathcal{U},\mathcal{W})}(u_i))$ crosses $\mathcal{R}_{s'_\alpha}(s_\alpha, u)$ and $\mathcal{R}(s_i, u_{i-1})$ crosses $\mathcal{R}(s_\alpha, u)$. We first prove that $\mathcal{R}(s'_i, W_{cv(\mathcal{U},\mathcal{W})}(u_i))$ intersects $\mathcal{R}_{s'_\alpha}(s_\alpha, u)$. Let $W_{cv(\mathcal{U},\mathcal{W})}(u_i) = w_h$. For $M^* \leq h \leq M$, it can be easily shown by induction that w_h is located above $\mathcal{R}(t'_{M^*}, w_M)$. Moreover, it is simple to see that s'_i must lie below t'_{M^*}. Then, knowing that $\mathcal{R}(t'_{M^*}, w_M)$ crosses $\mathcal{R}_{s'_\alpha}(s_\alpha, u)$ implies that $\mathcal{R}(s'_i, w_h)$ intersects $\mathcal{R}_{s'_\alpha}(s_\alpha, u)$ as well. From the fact that w_{M^*} lies inside $\mathcal{S}$, it can also be shown by induction that w_h for $h < M^*$ lies inside $\mathcal{S}$ which means that $\mathcal{R}(s'_i, w_h)$ crosses $\mathcal{R}_{s'_\alpha}(s_\alpha, u)$.

To complete the proof, we prove by induction on i that $\mathcal{R}(s_i, u_{i-1})$ crosses $\mathcal{R}(s_\alpha, u)$. It is clear that $s_{\nu+1}$ is located above s_α which means that $\mathcal{R}(s_{\nu+1}, u_\nu)$ intersects $\mathcal{R}(s_\alpha, u)$. From the previous paragraph we know that $\mathcal{R}_{s'_\alpha}(s_\alpha, u)$ intersects $\mathcal{R}(s'_{\nu+1}, W_{cv(\mathcal{U},\mathcal{W})}(u_{\nu+1}))$. Therefore, $\mathcal{R}(s_{\nu+1}, u_\nu)$ and $\mathcal{R}(s'_{\nu+1}, W_{cv(\mathcal{U},\mathcal{W})}(u_{\nu+1}))$ will intersect at a point within $\mathcal{S}$. Since we put $u_{\nu+1}$ at this intersection point, as the induction step, assume that u_{i-1} lies inside $\mathcal{S}$ where $i > \nu + 1$. Then, $\mathcal{R}(s_i, u_{i-1})$ intersects $\mathcal{R}(s_\alpha, u)$. □

5 Analysis

In previous sections, we proved several properties on the visibility graph of a pseudo-triangle and proposed an algorithm that constructs a pseudo-triangle for a given pair of visibility graph $\mathcal{G}(V, E)$ and Hamiltonian cycle $\mathcal{H}$ when this pair supports these properties. In this section, we analyze the time complexity of algorithms required to check these properties and the running time of the reconstruction algorithm.

We assume that we have the adjacency matrix of $\mathcal{G}$. Otherwise, we can obtain this matrix from adjacency list representation of the graph in $\mathcal{O}(n^2)$. To check Property 1, we need a linear time trace on vertices of $\mathcal{G}$ according to their order in $\mathcal{H}$. This is done in $\mathcal{O}(n)$ time. If two corners are identified in this way, the existence of a tower polygon corresponding to the pair of $\mathcal{G}$ and $\mathcal{H}$ can also be verified in linear time [3]. Property 2 can be verified in $\mathcal{O}(|E|)$ by a simple trace of the adjacency matrix or can be verified in $\mathcal{O}(n^2)$ using the adjacency list representation. In order to verify the rest of the properties, it is required to know the visibile subchains from each vertex. These subchains are obtained as by-products when checking Property 2. Having these subchains for each vertex, Property 3 can be verified in $\mathcal{O}(n)$. For any pair of side-chains $\mathcal{U}$ and $\mathcal{V}$ and a vertex $v \in \mathcal{V}$, let $\mathrm{T}_{\mathcal{U}}[v]$ be the farthest vertex of $\mathcal{U}$ from $cv(\mathcal{U}, \mathcal{V})$ which is visible to a vertex in $[cv(\mathcal{U}, \mathcal{V}), \ldots, v]$. Similarly, let $\mathrm{B}_{\mathcal{U}}[v]$ be the farthest vertex of $\mathcal{U}$ from $cv(\mathcal{U}, \mathcal{V})$ which is visible to a vertex in $[v, \ldots, cv(\mathcal{V}, \mathcal{W})]$. The arrays $\mathrm{B}_{\mathcal{U}}, \mathrm{B}_{\mathcal{V}}, \mathrm{B}_{\mathcal{W}}, \mathrm{T}_{\mathcal{U}}, \mathrm{T}_{\mathcal{V}}$, and $\mathrm{T}_{\mathcal{W}}$ are computed in $\mathcal{O}(n)$ having the visible subchains of all vertices by an aggregate-like trace on vertices of each side-chain in both direction. Property 4 can be checked in $\mathcal{O}(n)$: In this property, there are two different edges that leads to the existence of another edge. However, this property can be written as "for each vertex $u \in \mathcal{U}$, $\mathrm{T}_{\mathcal{U}}[v^{-1}_{cv(\mathcal{U},\mathcal{V})}(u)]$ must strictly lie above u" which can be verified in $\mathcal{O}(n)$. To verify Property 5, we must check that for any edge (u, v) when none of the edges $(u^{+1}_{cv(\mathcal{U},\mathcal{V})}, v)$ and $(u, v^{+1}_{cv(\mathcal{U},\mathcal{V})})$ exists, $\mathrm{B}_{\mathcal{U}}[v]$ must strictly lie above u.

This can be examined in $\mathcal{O}(|E|)$. In properties 6, 8, and 9, for each pair $(\mathcal{U}, \mathcal{V})$ of side-chains, we must find all pairs of visible vertices $(u \in \mathcal{U}, v \in \mathcal{V})$ such that $(u^{+1}_{cv(\mathcal{U},\mathcal{V})}, v)$ and $(u, v^{+1}_{cv(\mathcal{U},\mathcal{V})})$ are invisible. Having the adjacency matrix, this is done in $\mathcal{O}(|E|)$ by a Brute-Force algorithm. Then, examing properties 6, 8, and 9 is done in a constant time for each obtained pair. Checking properties 7 and 11 needs $\mathcal{O}(n)$ and examining Property 10 needs constant time. Therefore, all properties can be verified in $\mathcal{O}(n^2)$.

To complete the analysis, we compute the running time of the reconstruction algorithm presented in Section 4. Assume that $\mathcal{G}$ satisfies all of the properties introduced in Section 3 and we know the visible subchains of each vertex according to their order in $\mathcal{H}$. The side-chains of the target pseudo-triangle are identified in linear time according to the algorithm described in the proof of Lemma 1. Reconstructing $\mathcal{X}$ is done using *Colley*'s algorithm whose running time is linear in terms of the number of edges in the visibility graph reduced to $\mathcal{X}$. To reconstruct $\mathcal{Y}$, the algorithm needs to determine the floating edges of $\mathcal{U}$ and $\mathcal{V}$ which can be done in constant time. Computing the $\mathscr{W}$-type regions (for each vertex $w_i \in \mathcal{W}$) and determining the vertices w_{N^*} and w_{M^*} needs $\mathcal{O}(n)$ time. If the conditions of Lemma 3 are not satisfied, the floating edges of $\mathcal{U}$ and $\mathcal{V}$ must be extended which is done in $\mathcal{O}(1)$: A lower bound for the increase in floating edges can be computed by using *Thales*' theorem and trigonometric functions. Locating each vertex of $\mathcal{Y}$ is also done in constant time. Finally, placing each vertex of $\mathcal{Z}$ and $\mathcal{Z}'$ takes constant time, as well. Therefore, the total running time of the algorithm is $\mathcal{O}(n^2)$. We can combine all results as:

Theorem 2 *The visibility graph and the boundary vertices of a pseudo-triangle satisfy properties 1 to 11, and conversely, for any pair of graph $\mathcal{G}$ and Hamiltonian cycle $\mathcal{H}$ supporting these properties, there is a pseudo-triangle $\mathcal{P}$ whose visibility graph and boundary vertices are respectively isomorphic to $\mathcal{G}$ and $\mathcal{H}$. Checking these properties and reconstructing such a polygon can be done in $\mathcal{O}(n^2)$.*

6 Conclusion

In this paper, we considered properties of the visibility graph of a pseudo-triangle and obtained a set of necessary and sufficient conditions that such graphs must have. Then, we propose an algorithm to reconstruct a polygon from a given visibility graph which supports these properties. This characterizing and reconstructing problem has a long history and it seems that there is still a long way to be completed for all polygons.

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